Journal of Differential Equations **180**, 65–88 (2002)
doi:10.1006/jdeq.2001.4057, available online at http://www.idealibrary.com on IDEAL®

On the Multiplicity of the Solutions of the Equation $-\Delta u = \lambda \cdot f(u)$

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Received May 4, 2000; revised October 27, 2000

Let Ω be a bounded open subset of $\mathbb{R}^n$. We discuss conditions on the continuous function $f: \mathbb{R} \to \mathbb{R}$ ensuring that the problem $-\Delta u = \lambda \cdot f(u)$ admits unboundedly many weak solutions in $H_0^1(\Omega)$ for any positive real λ.

1. INTRODUCTION AND NOTATIONS

We are mainly concerned in this paper by the following nonlinear eigenvalue problem: we look at weak solutions in $H_0^1(\Omega)$ for the problem $-\Delta u = \lambda \cdot f(u)$, where Ω is a bounded open subset of the euclidean space $\mathbb{R}^n$, $\lambda > 0$ and $f: \mathbb{R} \to \mathbb{R}$ continuous. It is a well-known fact that this problem is closely related to the following variational problem: Find critical points in $H_0^1(\Omega)$ for the functional

$$\Phi_{\lambda'}: u \mapsto \int_\Omega F \circ u(x)\, dx - \lambda' \int_\Omega |\nabla u|^2\, dx,$$

where F is the primitive of f vanishing at 0 and $\lambda' = \frac{1}{2\lambda}$.

More generally, if f is a measurable function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ which is continuous with respect to the second variable, we can look at weak solutions for the equation $-\Delta u = \lambda f(\cdot, u(\cdot))$ as critical points for the functional

$$\Phi_{\lambda'}: u \mapsto \int_\Omega (F(x, u(x)) - \lambda' |\nabla u(x)|^2)\, dx,$$

where $\lambda' = \frac{1}{2\lambda}$ and $F(x, t) = \int_0^t f(x, s)\, ds$.

0022-0396/02 $35.00

In order to give sense to the previous formulas, we assume that for every $x \in \Omega$ and every $t \in \mathbb{R}$

$$|f(x,t)| \leqslant a\,|t|^p + b(x),$$

where $1 < p < \frac{n+2}{n-2}$, $a > 0$, and b is a positive element of $L^{q'}(\Omega)$, for $q' = \frac{p+1}{p}$. So we have

$$|F(x,t)| \leqslant a\,\frac{|t|^q}{q} + |t|\,b(x)$$

with $2 < q = p+1 < \frac{2n}{n-2}$. Then $H_0^1(\Omega)$ embeds compactly in $L^q(\Omega)$ and the functional $J\colon u \mapsto \int_\Omega F(x, u(x))\,dx$ is sequentially weakly continuous. Moreover, J is Fréchet differentiable at u and for $h \in H_0^1(\Omega)$ we get

$$J'(u) \cdot h = \int_\Omega f(x, \circ\, u(x)) \cdot h(x)\,dx$$

so that $\nabla J(u) = f(\cdot, u(\cdot)) \in L^{q'}(\Omega)$. It follows that ∇J is continuous from $H_0^1(\Omega)$ into $L^{q'}(\Omega)$ and that Φ_λ is weakly upper semicontinuous on each closed ball $B(r)$ and hence attains there its maximum since closed balls in $H_0^1(\Omega)$ are weakly compact.

Clearly, if the maximum is attained at an interior point u of the ball $B(r)$, it is a local maximum, and hence a critical point of Φ_λ, and we get

$$\nabla J(u) - \lambda \nabla \|\cdot\|^2(u) = f(\cdot, u(\cdot)) + 2\lambda \cdot \Delta u = 0;$$

it is $-\Delta u = \frac{1}{2\lambda} f(\cdot, u(\cdot))$.

More precisely we look for conditions under which we can guarantee the set of weak solutions of this problem is unbounded. In [4], Ricceri introduced tools for the study of local minima of a functional of the type $\Phi + \lambda\Psi$, where Φ and Ψ are weakly l.s.c. functionals on a reflexive Banach space. In the context of our problem $\Phi(u) = -\int_\Omega F(x, u(x))\,dx$ and $\Psi(u) = \int_\Omega |\nabla u(x)|^2\,dx$, and the boundedness of the set of the solutions is controlled by the behaviour of the function

$$Q(r) = \inf_{\|u\| < r} \frac{\sup_{\|v\| < r} J(v) - J(u)}{r^2 - \|u\|^2}.$$

On the other hand, Omari and Zanolin proved in [3] that the set of solutions of a similar problem is unbounded using convenient hypotheses on the potential F, and we refer to their paper for more details and for references on earlier works. Other important results about multiple solutions can be found in the paper [1] by Bartsch and Willem.

This paper is divided into three sections. In the first one we answer a conjecture of Ricceri. In the case where $F(t) = |t|^q$, the homogeneity of the function J allows an easy computation for the function Q above. And it follows that the conditions for the existence of infinitely many local maximum type solutions of the problem $-\Delta u = \lambda' f(u)$ cannot be fulfilled. But if we add a small perturbation to the previous function F, the computation of Q becomes very difficult. Nevertheless Ricceri conjectured that the result concerning the multiplicity of the solutions was unchanged, and we prove that this is the case.

In the next two sections we restrict ourselves to the autonomous case in which we consider only the case where the function $(x, t) \mapsto F(x, t)$ depends only on t.

In the first of these we study the case where the function F is not monotone on any neighborhood of $+\infty$ and prove that under extra mild conditions the set of solutions is unbounded.

Finally, in the last section, we show that the oscillatory behavior described in the previous section (with even a stronger property) is generic in some sense. Precisely, the set of those functions having this property is a comeager set in a Banach space of functions.

This work was performed in July 1999 during a stay of the author in Italy at the University of Catania, supported by the C.N.R. I am very grateful to Professor B. Ricceri for his stimulating questions.

2. A CONJECTURE OF B. RICCERI

The aim of this section is to prove that for a function like $F: t \mapsto |t|^q$ for $q > 2$, the functional Φ_λ cannot have unboundedly local maxima for any $\lambda > 0$.

LEMMA 2.1.

(i) *If Φ_λ attains its maximum over the ball $B(r)$ at an interior point then $Q(r) \leqslant \lambda$.*

(ii) *If Φ_λ attains its maximum over the ball $B(r)$ on the sphere then $Q(r) \geqslant \lambda$.*

Proof. Notice that since J is weakly continuous on each ball and since the sphere is weakly dense in the ball we have:

$$\mu(r) := \sup\{J(u) \colon \|u\| = r\} = \sup\{J(u) \colon \|u\| < r\}.$$

We also have

$$Q(r)=\inf_{\|u\|<r}\frac{\sup_{\|v\|<r} J(v)-J(u)}{r^2-\|u\|^2}=\inf_{s<r}\frac{\mu(r)-\mu(s)}{r^2-s^2}.$$

If there is a u_0 such that $s=\|u_0\|<r$ and $\Phi_\lambda(u)\leqslant\Phi_\lambda(u_0)$ for $\|u\|<r$, we have, for $\|u\|=r$,

$$\Phi_\lambda(u)=J(u)-\lambda r^2\leqslant\Phi_\lambda(u_0)\leqslant\mu(s)-\lambda s^2;$$

hence

$$\mu(r)-\lambda r^2\leqslant\mu(s)-\lambda s^2$$

and $Q(r)<\lambda$.

If Φ_λ attains its maximum at some point u_0 of the sphere, we have, for $s<r$,

$$\mu(s)-\lambda s^2=\sup_{\|v\|=s}\Phi_\lambda(v)\leqslant\sup_{\|v\|\leqslant r}\Phi_\lambda(v)=\Phi_\lambda(u_0)\leqslant\mu(r)-\lambda r^2;$$

hence $Q(r)\geqslant\lambda$. ∎

Corollary 2.2. *If*

$$\liminf_{r\to\infty} Q(r)<\lambda<\limsup_{r\to\infty} Q(r)$$

there is a sequence (u_n) such that $\|u_n\|\to\infty$ and that each u_n is a local maximum of Φ_λ and hence satisfies $-\Delta u_n+\frac{1}{2\lambda}f\circ u_n=0$.

Proof. One can choose an increasing sequence (r_n) such that $r_{n+1}>r_n+1$ for each n and that $Q(r_{2n})<\lambda<Q(r_{2n+1})$. Since $Q(r_{2n-1})>\lambda$, the maximum of Φ_λ over the ball $B(r_{2n})$ cannot be attained in the ball $B(r_{2n-1})$, but since $Q(r_{2n})<\lambda$, it cannot be attained on the sphere $S(r_{2n})$. Then it is attained at some point u_n such that $r_{2n-1}\leqslant\|u_n\|<r_{2n}$, and it is a local maximum. ∎

Definition. We will say that f is of type LM_∞ if

$$\liminf_{r\to\infty} Q(r)=0 \qquad \text{and} \qquad \limsup_{r\to\infty} Q(r)=+\infty.$$

If f is of type LM_∞, there are unboundedly many critical points for Φ_λ for each positive real λ. It is quite unclear whether simple functions can be of type LM_∞. First of all, we show that it is not the case for the function $f: t \mapsto t \cdot |t|^{p-1}$ (with $1 < p < \frac{n+2}{n-2}$), even with a small perturbation. The following theorem answers a conjecture of Ricceri.

THEOREM 2.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $1 < p < \frac{n+2}{n-2}$, $F: \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function of class $\mathscr{C}^1$ with respect to the second variable, satisfying*

(i) $\xi \frac{\partial F}{\partial \xi}(x, \xi) \geqslant \varepsilon\, |\xi|^{p+1} - b(x)\, |\xi|$

(ii) $|\frac{\partial F}{\partial \xi}(x, \xi)| \leqslant a\, |\xi|^p + b(x)$

for some positive a and ε, and some positive function $b \in L^{q'}(\Omega)$ with $q' = \frac{p+1}{p}$. Then we have $\lim_{r \to \infty} Q(r) = +\infty$.

Notice first that, until F is replaced by $F_0: (x, t) \mapsto F(x, t) - F(x, 0)$, which adds a constant to J, we can assume $F(x, 0) = 0$. Then, with $f(x, \xi) = \frac{\partial F}{\partial \xi}(x, \xi)$, we have

$$F(x, t) = \int_0^1 t\xi \cdot f(x, t\xi)\, \frac{d\xi}{\xi} \geqslant \varepsilon\, \frac{|t|^{p+1}}{p+1} - |t|\, b(x)$$

and

$$|F(x, t)| \leqslant \int_0^1 |t\xi \cdot f(x, t\xi)|\, \frac{d\xi}{\xi} \leqslant a\, \frac{|t|^{p+1}}{p+1} + |t|\, b(x).$$

In order to prove the theorem, we begin by proving several lemmas.

Let us denote by u_0 a fixed nonnegative function of class $\mathscr{C}^1$ with compact support in Ω and norm 1 in $H_0^1(\Omega)$. We also denote $\|v\|_p = (\int_\Omega |v(x)|^p\, dx)^{1/p}$ for $p \geqslant 1$ and $v \in L^p(\Omega)$.

LEMMA 2.4. *For all $\lambda > 0$, there exist $\delta > 0$ and $R_0 > 0$ such that for every u in the unit sphere $S(1)$ of $H_0^1(\Omega)$ the following holds:*

$$t \geqslant R_0 \quad \text{and} \quad \|u\|_q < \delta \Rightarrow J(tu) - \lambda t^2 < \tfrac{1}{2}(J(tu_0) - \lambda t^2).$$

Proof. We have $F(x, \xi) \geqslant \varepsilon(|\xi|^q/q) - b(x)\, |\xi|$; hence, for $t \geqslant 0$,

$$J(tu_0) - \lambda t^2 \geqslant \int_\Omega \left(\varepsilon t^q\, \frac{|u_0(x)|^q}{q} - tb(x)\, u_0(x) \right) dx - \lambda t^2$$

$$\geqslant \frac{\varepsilon t^q}{q}\, \|u_0\|_q^q - t\, \|b\|_{q'}\, \|u_0\|_q - \lambda t^2$$

and similarly, since $|F(x, \xi)| \leqslant a(|\xi|^q/q) + b(x)\,|\xi|$,

$$|J(tu) - \lambda t^2| \leqslant \int_\Omega \left(at^q \frac{|u(x)|^q}{q} + tb(x)\,u(x) \right) dx + \lambda t^2$$
$$\leqslant \frac{at^q}{q}\,\|u\|_q^q + t\,\|b\|_{q'}\,\|u\|_q + \lambda t^2.$$

So, if $\|u\|_q < \delta$,

$$(J(tu_0) - \lambda t^2) - 2\,|J(tu) - \lambda t^2|$$
$$\geqslant \frac{t^q}{q}\,(\varepsilon\,\|u_0\|_q^q - 2a\delta^q) - (t\,\|b\|_{q'}\,(\|u_0\|_q + 2\delta) + 3\lambda t^2)$$

and if we choose δ in such a way that $4a\delta^q < \varepsilon\,\|u_0\|_q^q$, we get for a convenient a':

$$(J(tu_0) - \lambda t^2) - 2\,|J(tu) - \lambda t^2| \geqslant t^q \frac{\varepsilon}{2q}\,\|u_0\|_q^q - a'(t + t^2).$$

And since $q > 2$, there exists a $R_0 > 0$ such that $t^q \frac{\varepsilon}{2q}\,\|u_0\|_q^q > a'(t + t^2)$ for $t \geqslant R_0$. ∎

LEMMA 2.5. *For $\lambda > 0$ and $u \in S(1)$ the function $g_{\lambda, u}\colon t \mapsto J(tu) - \lambda t^2$ is derivable, and its derivative satisfies*

$$g'_{\lambda, u}(t) \geqslant \varepsilon t^{q-1}\,\|u\|_q^q - 2\lambda t - \|b\|_{q'}\,\|u\|_q$$

and

$$|g'_{\lambda, u}(t)| \leqslant at^{q-1}\,\|u\|_q^q + 2\lambda t + \|b\|_{q'}\,\|u\|_q.$$

Proof. Since $J(tu) = \int_\Omega F(x, tu(x))\,dx$ we get

$$\frac{d}{dt} J(tu) = \int_\Omega \frac{\partial F}{\partial \xi}(x, tu(x)) \cdot u(x)\,dx = \int_\Omega f(x, tu(x)) \cdot u(x)\,dx;$$

hence

$$\varepsilon t^{q-1}\,|u(x)|^q - b(x)\,|u(x)| \leqslant f(x, u(x)) \cdot u(x) \leqslant |f(x, u(x)) \cdot u(x)|$$
$$\leqslant at^{q-1}\,|u(x)|^q + b(x)\,|u(x)|$$

and

$$\varepsilon t^{q-1} \|u\|_q^q - \|b\|_{q'} \|u\|_q \leqslant \frac{d}{dt} J(tu) \leqslant \left| \frac{d}{dt} J(tu) \right| \leqslant a t^{q-1} \|u\|_q^q + \|b\|_{q'} \|u\|_q$$

and the result follows from these inequalities. ▮

LEMMA 2.6. *For each $\lambda > 0$ there is some $R \geqslant R_0$ such that for $u \in S(1)$ and $t \geqslant R$:*

$$J(tu) - \lambda t^2 \geqslant \tfrac{1}{2} (J(tu_0) - \lambda t^2) \Rightarrow g'_{\lambda, u}(t) > 0.$$

Proof. It follows from Lemma 2.4 that conditions $\|u\| = 1$, $t \geqslant R_0$ and $J(tu) - \lambda t^2 \geqslant \frac{1}{2} (J(tu_0) - \lambda t^2)$ imply $\|u\|_q \geqslant \delta$. Then by Lemma 2.5

$$\begin{aligned} g'_{\lambda, u}(t) &\geqslant \varepsilon t^{q-1} \|u\|_q^q - 2\lambda t - \|b\|_{q'} \|u\|_q \\ &\geqslant t^{q-1} \|u\|_q^q \left(\varepsilon - 2 \frac{\lambda}{\|u\|_q^q} t^{2-q} - \frac{\|b_{q'}\|}{\|u_q^{q-1}\|} t^{1-q} \right) \\ &\geqslant t^{q-1} \|u\|_q^q \left(\varepsilon - 2 \frac{\lambda}{\delta^q} t^{2-q} - \frac{\|b_{q'}\|}{\delta^{q-1}} t^{1-q} \right) \end{aligned}$$

and if $R \geqslant R_0$ is chosen large enough so that $(2\lambda)/(\delta^q R^{q-2}) + \|b_{q'}\|/(\delta^{q-1} R^{q-1}) \leqslant \frac{\varepsilon}{2}$, we get

$$g'_{\lambda, u}(t) \geqslant \frac{\delta^q \varepsilon}{2} t^{q-1} > 0$$

for $t \geqslant R$. ▮

LEMMA 2.7. *For $\lambda > 0$ and $\rho > 0$, there exists a constant $\beta > 0$ such that, for $u \in S(1)$, t, and t' in $[0, \rho]$, one has:*

$$|g_{\lambda, u}(t') - g_{\lambda, u}(t)| \leqslant \beta |t' - t|.$$

Proof. Since $q < \frac{2n}{n-2}$, $H_0^1(\Omega)$ embeds continuously into $L^q(\Omega)$. Thus there is a constant γ such that, for $u \in H_0^1(\Omega)$, $\|u\|_q \leqslant \gamma \|u\|$. Moreover by Lemma 2.5, one has

$$g'_{\lambda, u}(t)| \leqslant a t^{q-1} \|u\|_q^q + \|b\|_{q'} \|u\|_q + 2\lambda t.$$

Then for $0 < t < \rho$ and $\|u\| = 1$

$$|g'_{\lambda, u}(t)| \leqslant \beta := a \rho^{q-1} \gamma^q + \gamma \|b\|_{q'} + 2\lambda \rho$$

and the result follows from the mean value theorem. ▮

LEMMA 2.8. *For $\lambda > 0$ and $\rho > 0$ the function $r \mapsto \mu(r) - \lambda r^2$ is Lipschitz on $[0, \rho]$.*

Proof. From Lemma 2.7 we know that there is a $\beta > 0$ such that for t and s in $[0, \rho]$ and $u \in S(1)$:

$$g_{\lambda, u}(s) \leqslant g_{\lambda, u}(t) + \beta\,|s-t| \leqslant (\mu(t) - \lambda t^2) + \beta\,|s-t|.$$

Hence $\mu(s) - \lambda s^2 \leqslant (\mu(t) - \lambda t^2) + \beta\,|s-t|$ and this shows that the function: $r \mapsto \mu(r) - \lambda r^2$ is β-Lipschitz on $[0, \rho]$. ▮

For fixed λ, we consider the sets

$$T = \{r \in \mathbb{R}^+ : \mu(r) - \lambda r^2 = \sup_{s \leqslant r} (\mu(s) - \lambda s^2)\}$$

$$T' = \{r \in \mathbb{R}^+ : \forall s < r \ \mu(s) - \lambda s^2 < \mu(r) - \lambda r^2\}.$$

LEMMA 2.9. *The set T is closed in $\mathbb{R}^+$ and T' is unbounded.*

Proof. The function $r \mapsto \mu(r) - \lambda r^2$ is locally Lipschitz and hence continuous. Thus the set

$$T = \bigcap_{s \in \mathbb{R}^+} (\{r : \mu(r) - \lambda r^2 \geqslant \mu(s) - \lambda s^2\} \cup [0, s])$$

is closed.

Let $\theta > 0$. We show that T' meets $]\theta, +\infty[$. Since μ is continuous, we have:

$$A_\theta := \sup_{s \leqslant \theta} (\mu(s) - \lambda s^2) < +\infty.$$

From the proof of Lemma 2.4, we know that

$$\begin{aligned}(\mu(r) - \lambda r^2) - r^2 &\geqslant g_{\lambda, u_0}(r) - r^2 = J(ru_0) - (\lambda+1)\, r^2 \\ &\geqslant r^q \frac{\varepsilon}{q} \|u_0\|_q^q - r\,\|u_0\|_q - (\lambda+1)\, r^2.\end{aligned}$$

Thus there exists an R' such that $g_{\lambda, u_0}(r) - r^2 > 0$ for $r \geqslant R'$; that is $\mu(r) - \lambda r^2 > r^2$ for $r \geqslant R'$.

Then with $r > \max(R', \sqrt{A_\theta})$ we get $M := \mu(r) - \lambda r^2 > A_\theta$. And the closed set $F = \{s \in \mathbb{R}^+ : \mu(s) - \lambda s^2 \geqslant M\}$, which is nonempty since $r \in F$, has a least element r_0. Since $A_\theta < M \leqslant \mu(r_0) - \lambda r_0^2$ we get $r_0 > \theta$. And for every $s < r_0$ we have $s_0 \notin F$; hence $\mu(s) - \lambda s^2 < M \leqslant \mu(r_0) - \lambda r_0^2$ and $r_0 \in T'$. Thus $r_0 \in T' \cap]\theta, +\infty[$.

Since θ is arbitrary, this shows that T' is unbounded. ▮

LEMMA 2.10. *The set T' contains a half-line* $[R_1, +\infty[$.

Proof. Let λ be fixed. By Lemmas 2.4 and 2.6, there is a $R > 0$ such that for $r \geqslant R$ and $\|u\| = 1$ we have $g_{\lambda, u_0}(r) > 0$ and

$$g_{\lambda, u}(r) > \tfrac{1}{2}\, g_{\lambda, u}(r_0), \quad \Rightarrow g'_{\lambda, u}(r) > 0.$$

Moreover, by Lemma 2.9 there is an $R_1 > R$ in T'. We show that $[R_1, +\infty[\subset T'$. Assume by contradiction that $r \notin T'$ for some $r > R_1$. Then the function $u \mapsto J(u) - \lambda \|u\|^2$ attains its maximum α on the closed ball $B(r)$ at some point u^* with $s = \|u^*\| < r$.

Since $R_1 < r$, we have

$$\mu(R_1) - \lambda R_1^2 = \sup_{\|u\| = R_1} J(u) - \lambda \|u\|^2 \leqslant \alpha = J(u^*) - \lambda \|u^*\|^2 \leqslant \mu(s) - \lambda s^2;$$

hence $s \geqslant R_1 \geqslant R$. And for $u^{**} = \frac{1}{s} u^*$, we get

$$g_{\lambda, u^{**}}(s) = \alpha \geqslant g_{\lambda, u_0}(s) > 0$$

which implies $g_{\lambda, u^{**}}(s) > \frac{1}{2}\, g_{\lambda, u_0}(s)$; hence $g'_{\lambda, u^{**}}(s) > 0$. We conclude that there is s' with $s < s' < r$ such that

$$g_{\lambda, u^{**}}(s') - g_{\lambda, u^{**}}(s) > \tfrac{1}{2}\,(s' - s)\, g_{\lambda, u^{**}}(s) > 0$$

and finally

$$\alpha = g_{\lambda, u^{**}}(s) < g_{\lambda, u^{**}}(s') \leqslant \sup_{\|u\| \leqslant r} J(u) - \lambda \|u\|^2 = \alpha$$

and this contradiction completes the proof. ▮

Proof of Theorem 2.3. It follows from the previous lemma that for every $\lambda > 0$ there is an $R_1 > 0$ such that

$$r \geqslant R_1 \qquad \text{and} \qquad s < r \Rightarrow \mu(s) - \lambda s^2 < \mu(r) - \lambda r^2.$$

Thus by Lemma 2.1 $Q(r) \geqslant \lambda$ for $r \geqslant R_1$. And since λ is arbitrary we get

$$\lim_{r \to \infty} Q(r) = +\infty. \quad ▮$$

3. OSCILLATING FUNCTIONS

A consequence of Theorem 2.3 is that in order to find functions of type LM_∞ we have to look at more oscillating functions, like in the paper [3] by Omari and Zanolin.

THEOREM 3.1. *Assume* $1 < p < \frac{n+2}{n-2}$ *and let* $F\colon \mathbb{R} \to \mathbb{R}$ *be of class* $\mathscr{C}^1$ *and satisfy:*

(i) $|F'(t)| \leqslant a(1+|t|^p)$.

(ii) *There exists a constant* $M > 0$ *such that for every* $\rho > 0$ *there exists a* $t > 0$ *satisfying* $F(t) \geqslant \rho(1+t^2)$ *and* $F(s) \geqslant -M \cdot F(t)$ *for* $0 \leqslant s \leqslant t$.

(iii) $\sup\{t\colon F'(t) < 0\} = +\infty$.

(iv) $\inf\{t\colon F'(t) > 0\} < 0$.

Then there are unboundedly many local maxima of Φ_λ *for every positive* λ.

Put $f = F'$. By hypothesis there is an increasing sequence $(\theta_n)_{n \geqslant 1}$ converging to $+\infty$ such that $f(\theta_n) < 0$. Thus we can find a sequence θ'_n such that $\theta_n < \theta'_n < \theta_{n+1}$ and

$$\forall t \in \,]\theta_n, \theta'_n[\qquad F(t) < F(\theta_n).$$

Similarly there are θ_0 and θ'_0 such that $\theta'_0 < \theta_0 < 0$ and

$$\forall t \in \,]\theta'_0, \theta_0[\qquad F(t) < F(\theta_0).$$

Consider for $n \geqslant 1$ the set

$$E_n = \{u \in H^1_0(\Omega) : \forall x \in \Omega \; \theta'_0 \leqslant u(x) \leqslant \theta'_n\}.$$

Since $H^1_0(\Omega)$ embeds in $L^q(\Omega)$, the mapping $u \mapsto F \circ u$ is continuous from $H^1_0(\Omega)$ into $L^1(\Omega)$. Thus the set E_n is closed in $H^1_0(\Omega)$ and clearly convex. Hence E_n is weakly closed in $H^1_0(\Omega)$. Let us denote by M_n the supremum of F on the compact interval $[\theta'_0, \theta'_n]$.

LEMMA 3.2. *The functional* Φ_λ *is bounded from above on* E_n *and attains its maximum.*

Proof. For $u \in E_n$ and $x \in \Omega$ we have $F \circ u(x) \leqslant M_n$; hence

$$\Phi_\lambda(u) \leqslant J(u) = \int_\Omega F \circ u(x)\, dx \leqslant M_n\, |\Omega|.$$

Then, with $\alpha_n = \sup_{u \in E_n} \Phi_\lambda(u)$, the set

$$A_{n,k} = \{u \in E_n : \Phi_\lambda(u) \geqslant \alpha_n - 2^{-k}\}$$

is bounded: indeed, for $u \in A_{n,k}$ we have

$$\|u\|^2 = \frac{1}{\lambda}(J(u) - \Phi_\lambda(u)) \leqslant \frac{1}{\lambda}(M_n\,|\Omega| - \alpha_n + 2^{-k})$$

and since Φ_λ is sequentially weakly u.s.c. the sets $A_{n,k}$ are bounded and weakly closed and hence weakly compact. They are nonempty by definition of α_n. Thus their intersection K_n is nonempty, and Φ_λ attains its maximum α_n at each point u^* of K_n. ∎

It is worth noticing that the set K_n is norm-compact in $H_0^1(\Omega)$. Indeed, if (u_j) is a sequence in K_n there is a subsequence $(u_j)_{j \in I}$ which converges weakly to some $u \in K_n$. Then

$$\|u_j\|^2 = \frac{1}{\lambda}(J(u_j) - \Phi_\lambda(u_j)) = \frac{1}{\lambda}(J(u_j) - \alpha_n)$$

and since J is weakly continuous this converges (for $j \in I$) to $\frac{1}{\lambda}(J(u) - \alpha_n) = \|u\|^2$. Then

$$\|u - u_j\|^2 = \|u_j\|^2 + \|u\|^2 - 2\langle u, u_j\rangle \to \|u\|^2 + \|u\|^2 - 2\langle u, u\rangle = 0.$$

So the sequence $(u_j)_{j \in I}$ converges strongly to u.

LEMMA 3.3. *The sequence α_n grows to infinity.*

Proof. Let T be fixed and choose ρ_0 such that $\rho_0\,|\Omega| \geqslant 2T$. For $\varepsilon > 0$, denote by Ω_ε the open subset $\{x \in \mathbb{R}^n : d(x, \mathbb{R}^n \setminus \Omega) < \varepsilon\}$ of Ω. We have $\bigcap_{\varepsilon > 0} \Omega_\varepsilon = \varnothing$; thus we can find some $\varepsilon > 0$ such that

$$|\Omega_\varepsilon| < \frac{1}{2(M+2)}\,|\Omega|.$$

Then, there is some $\rho \geqslant \rho_0$ such that $\rho\varepsilon^2 > 1$, some β such that $F(\beta) > \rho(1+\beta^2)$ and $F(s) \geqslant -M \cdot F(\beta)$ for $0 \leqslant s \leqslant \beta$, and some integer n such that $\beta \leqslant \theta_n$. We put

$$u_0(x) = \beta \inf(1, \varepsilon^{-1} \cdot d(x, \mathbb{R}^n \setminus \Omega)).$$

It is easy to check that u_0 is $\frac{\beta}{\varepsilon}$-Lipschitz and hence that $|\nabla u_0| \leqslant \frac{\beta}{\varepsilon}$ a.e. Then $u_0 \in E_n$ and $u_0(x) < \beta$ for $x \in \Omega_\varepsilon$. Hence,

$$\begin{aligned} J(u_0) &= \int_{\Omega_\varepsilon} F \circ u_0(x)\, dx + \int_{\Omega \setminus \Omega_\varepsilon} F \circ u_0(x)\, dx \\ &\geqslant -M \cdot F(\beta)\, |\Omega_\varepsilon| + F(\beta)(|\Omega| - |\Omega_\varepsilon|) \\ &\geqslant F(\beta)(|\Omega| - M(+1)\, |\Omega_\varepsilon|) \geqslant \rho(1+\beta^2)\, |\Omega| - (M+1)\, \rho(1+\beta^2)\, |\Omega_\varepsilon|. \end{aligned}$$

Moreover we have $|\nabla u_0(x)| = 0$ a.e. on $\Omega \setminus \Omega_\varepsilon$ and $|\nabla u_0(x)| \leqslant \frac{\beta}{\varepsilon}$ a.e. on Ω_ε; hence

$$\|u_0\|^2 = \int_\Omega |\nabla u_0(x)|^2\, dx \leqslant \left(\frac{\beta}{\varepsilon}\right)^2 |\Omega_\varepsilon|.$$

We conclude that

$$\begin{aligned} \Phi_\rho(u_0) &\geqslant \rho(1+\beta^2)\left(|\Omega| - \left(M + 1 + \frac{1}{p\varepsilon^2}\right) |\Omega_\varepsilon|\right) \\ &\geqslant \rho(1+\beta^2)(|\Omega| - (M+2)\, |\Omega_\varepsilon|) \geqslant \frac{\rho(1+\beta^2)}{2}\, |\Omega| \\ &\geqslant \frac{\rho_0}{2}\, |\Omega| \geqslant T. \end{aligned}$$

Then we have

$$\alpha_n = \sup_{u \in E_n} \Phi_\lambda(u) \geqslant \Phi_\lambda(u_0) \geqslant T$$

and this shows that $\sup_n \alpha_n = +\infty$. ∎

The following well-known superposition lemma can be found in [2]:

LEMMA 3.4 (Marcus and Mizel). *Let Ω be an open subset of $\mathbb{R}^n$, $1 \leqslant p < \infty$, and $\varphi: \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping satisfying $\varphi(0) = 0$. Then $\varphi \circ u$ belongs to $W^{1,p}$ for any $u \in W^{1,p}(\Omega)$. Furthermore the mapping $T: u \mapsto \varphi \circ u$ is continuous from $W^{1,p}$ into itself.*

In particular, if φ is piecewise affine and vanishes at 0 the mapping $u \mapsto \varphi \circ u$ is continuous from $H_0^1(\Omega)$ to $H_0^1(\Omega)$.

LEMMA 3.5. *If u^* belongs to K_n then $\theta_0 \leqslant u^*(x) \leqslant \theta_n$ a.e. on Ω.*

Let $h\colon \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$h(t) = \begin{cases} \theta_0 & \text{if } \ t \leqslant \theta_0 \\ t & \text{if } \ \theta_0 \leqslant t \leqslant \theta_n \\ \theta_n & \text{if } \ t \geqslant \theta_n. \end{cases}$$

Then by Lemma 3.4 $v^* = h \circ u^*$ belongs to E_n too and if X denotes the measurable set $\{x \in \Omega : u^*(x) \notin [\theta_0, \theta_n]\}$ we have for $x \in X$:

— either $\theta_0' \leqslant u^*(x) < \theta_0$ and $F \circ v^*(x) = F(\theta_0) > F \circ u^*(x)$
— or $\theta_n \leqslant u^*(x) < \theta_n'$ and $F \circ v^*(x) = F(\theta_n) > F \circ u^*(x)$.

Then $F \circ v^* > F \circ u^*$ a.e. on X and $F \circ v^* = F \circ u^*$ a.e. on $\Omega \setminus X$. Moreover we have $\nabla v^* = 0$ a.e. on X and $\nabla v^* = \nabla u^*$ a.e. on $\Omega \setminus X$. It follows that

$$\Phi_\lambda(v^*) - \Phi_\lambda(u^*) = \int_X (F \circ v^*(x) - F \circ u^*(x))\, dx + \lambda \int_X |\nabla u^*(x)|^2\, dx$$

and since $u^* \in K_n$ and $v^* \in E_n$, we have $\Phi_\lambda(v^*) - \Phi_\lambda(u^*) \leqslant 0$. This implies:

$$\int_X (F \circ v^*(x) - F \circ u^*(x))\, dx = \int_X |\nabla u^*(x)|^2\, dx = 0.$$

Since $F \circ v^*(x) > F \circ u^*(x)$ a.e. on X, this implies in turn that X is a null set; hence

$$\|v^* - u^*\|^2 = \int_X |\nabla u^*(x)|^2\, dx = 0$$

and $u^* = v^*$, which means $u^*(x) = v^*(x) \in [\theta_n, \theta_n]$ for almost all x in Ω. ∎

LEMMA 3.6. *Every element of K_n is a local maximum of Φ_λ.*

Proof. Let $u^* \in K_n$ and h be as above. Then the mapping $T\colon u \mapsto h \circ u$ is continuous from $H_0^1(\Omega)$ to $H_0^1(\Omega)$. Let

$$G(\xi) = F(\xi) - F \circ h(\xi).$$

For $u \in H_0^1(\Omega)$, we have $Tu \in E_n$; hence $\Phi_\lambda(Tu) \leqslant \Phi_\lambda(u^*)$ and

$$\Phi_\lambda(u) - \Phi_\lambda(Tu) = \int_\Omega G \circ u(x)\, dx - \lambda \int_\Omega (|\nabla u(x)|^2 - |\nabla(Tu)(x)|^2)\, dx.$$

As in the previous lemma, if X denotes the set $\{x \in \Omega : u(x) \notin [\theta_0, \theta_n]\}$, we have $\nabla u(x) = \nabla(Tu)(x)$ a.e. on $\Omega \backslash X$ and $\nabla(Tu)(x) = 0$ a.e. on X; hence

$$\int_\Omega (|\nabla u(x)|^2 - |\nabla(Tu)(x)|^2)\, dx = \int_X |\nabla u(x)|^2\, dx$$

$$= \int_\Omega |\nabla(u - Tu)(x)|^2\, dx = \|u - Tu\|^2.$$

Since by condition (i) we have

$$|F'(t)| \leqslant a(1 + |t|^p)$$

for all $t \in \mathbb{R}$, we have

$$|F(\xi)| = \left| F(0) + \int_0^\xi F'(t)\, dt \right| \leqslant |F(0)| + \int_0^{|\xi|} a(1 + t^p)\, dt \leqslant a_1(1 + |\xi|^{p+1})$$

for some constant a_1.

Then the function G satisfies:

$$G(\xi) \begin{cases} = 0 & \text{if} \quad \theta_0 \leqslant \xi \leqslant \theta_n \\ < 0 & \text{if} \quad \theta'_0 < \xi < \theta_0 \text{ or } \theta_n < \xi < \theta'_n \\ \leqslant 2a_1(1 + |\xi|^{p+1}) & \text{for all} \quad \xi \in \mathbb{R}. \end{cases}$$

Thus

$$C = \sup \left\{ \frac{|G(\xi)|}{|\xi - h(\xi)|^{p+1}} : \xi \notin]\theta'_0, \theta'_n[\right\} < +\infty$$

and it follows that for every $\xi \in \mathbb{R}$ the following holds:

$$G(\xi) \leqslant C\, |\xi - h(\xi)|^{p+1}.$$

Since $q = p + 1 < \frac{2n}{n-2}$, $H_0^1(\Omega)$ embeds continuously in $L^q(\Omega)$ and there is some γ such that $\|u\|_q \leqslant \gamma\, \|u\|$ for every $u \in H_0^1(\Omega)$. Hence

$$\int_\Omega G \circ u(x)\, dx \leqslant C \int_\Omega |(u - Tu)(x)|^q\, dx = C\, \|u - Tu\|_q^q \leqslant C\gamma^q\, \|u - Tu\|^q$$

and

$$\Phi_\lambda(u) - \Phi_\lambda(Tu) \leqslant C\gamma^q\, \|u - Tu\|^q - \lambda\, \|u - Tu\|^2$$

$$\Phi_\lambda(u) \leqslant \Phi_\lambda(u^*) - \lambda\, \|u - Tu\|^2 \left(1 - \frac{C\gamma^q}{\lambda}\, \|u - Tu\|^{p-1} \right).$$

Since T is continuous and $Tu^* = u^*$, there is some $\alpha > 0$ such that

$$\|u - u^*\| < \alpha \Rightarrow \|u - Tu\|^{p-1} < \frac{\lambda}{2C\gamma^q}.$$

Then for $u \in B(u^*, \alpha)$ we get

$$\Phi_\lambda(u) \leqslant \Phi_\lambda(u^*) - \frac{\lambda}{2}\|u - Tu\|^2 \leqslant \Phi_\lambda(u^*)$$

and this shows that Φ_λ has a local maximum at u^*. ∎

Proof of Theorem 3.1. Let $R > 0$. We have to prove the existence of some local maximum of Φ_λ whose norm is larger than R. As in the previous lemma we have $|F(\xi)| \leqslant a_1(1 + |\xi|^{p+1})$. If $u \in H_0^1(\Omega)$ has norm less than R, we have, for $q = p + 1$:

$$J(u) = \int_\Omega F \circ u(x)\, dx \leqslant \int_\Omega a_1(1 + |u(x)|^q)\, dx$$
$$\leqslant a_1(|\Omega| + \|u\|_q^q) \leqslant a(|\Omega| + \gamma^q R^q).$$

Hence

$$\Phi_\lambda(u) = J(u) - \lambda\, \|u\|^2 \leqslant J(u) \leqslant C_R := a(|\Omega| + \gamma^q R^q).$$

Since the sequence (α_n) goes to the infinity, there is some n such that $\alpha_n > C_R$. Then no element of K_n can belong to the ball $B(R)$. Thus each point of K_n is a local maximum of Φ_λ and has a norm larger than R. ∎

COROLLARY 3.7. *Assume $p < \frac{n+2}{n-2}$ and $2 < q \leqslant p + 1$. Let $F\colon \mathbb{R} \to \mathbb{R}$ be of class $\mathscr{C}^1$ and satisfy:*

(i) $|F'(t)| \leqslant a(1 + |t|^p)$
(ii) $|F(t)| \leqslant a(1 + t^q)$ *for* $t \geqslant 0$ *and* $\limsup_{t \to +\infty} t^{-q}F(t) > 0$
(iii) $\sup\{t\colon F'(t) < 0\} = +\infty$
(iv) $\inf\{t\colon F'(t) > 0\} < 0$.

Then there are unboundedly many local maxima of Φ_λ for every positive λ.

Proof. We have only to prove that condition (ii) of Theorem 3.1 is fulfilled. Choose $\eta > 0$ such that $\eta < \limsup_{t \to +\infty} t^{-q}\, F(t)$ and then put

$M = \frac{2a}{\eta}$. If ρ is chosen, there is some $t_0 > 1$ such that $\eta t_0^q > \rho(1+t_0^2)$, hence a $\beta > t_0$ such that

$$F(\beta) > \eta\beta^q = \left(\frac{\beta}{t_0}\right)^q \eta t_0^q > \left(\frac{\beta}{t_0}\right)^2 \rho(1+t_0^2) > \rho(1+\beta^2),$$

while for $0 \leqslant s \leqslant \beta$ we have

$$F(s) \geqslant -|F(s)| \geqslant -a(1+s^q) \geqslant -a(1+\beta^q) \geqslant -\frac{M}{2}\eta(1+\beta^q)$$
$$\geqslant -M\eta\beta^q \geqslant -M\cdot F(\beta)$$

and this completes the proof. ∎

Corollary 3.8. *Assume* $1 < p < \frac{n+2}{n-2}$. *Let* $F\colon \mathbb{R} \to \mathbb{R}$ *be of class* $\mathscr{C}^1$ *and satisfy:*

(i) $|F'(t)| \leqslant a(1+|t|^p)$
(ii″) $\liminf_{t\to+\infty} t^{-2}F(t) > -\infty$ *and* $\limsup_{t\to+\infty} t^{-2}F(t) = +\infty$
(iii) $\sup\{t\colon F'(t) < 0\} = +\infty$
(iv) $\inf\{t\colon F'(t) > 0\} < 0$.

Then there are unboundedly many local maxima of Φ_λ *for every positive* λ.

As above we have only to prove that condition (ii) of Theorem 3.1 holds. Let $M > 0$ such that $\inf_{t\geqslant 0} F(t)/(1+t^2) > -M$. Let $\rho > 1$. There is some $\beta > 1$ such that $\beta^{-2}F(\beta) > 2\rho$. Then we have

$$F(\beta) \geqslant 2\rho\beta^2 \geqslant \rho(1+\beta^2)$$

and, for $0 \leqslant s \leqslant \beta$,

$$F(s) \geqslant -M(1+s^2) \geqslant -M\rho(1+s^2) \geqslant -M\rho(1+\beta^2) \geqslant -M\cdot F(\beta)$$

and this completes the proof. ∎

Example 3.9. Assume $2 < q < q+\delta < \frac{2n}{n-2}$ and φ is a $\mathscr{C}^1$ periodic non-constant function on $\mathbb{R}$. Then if φ is not everywhere negative the function

$$F\colon \xi \mapsto |\xi|^q\,\varphi(|\xi|^\delta)$$

satisfies the condition of Corollary 3.7.

We have, for $\xi \neq 0$,

$$f(\xi) = F'(\xi) = \xi\,|\xi|^{q+\delta-2}\left(\delta\varphi'(|\xi|^{\delta}) + \frac{q}{|\xi|^{\delta}}\,\varphi(|\xi|^{\delta})\right) = O(|\xi|^{q+\delta-1}).$$

Hence condition (i) is satisfied. Let T be the period of φ. Since φ is not constant and

$$\int_0^T \varphi'(t)\,dt = \varphi(T) - \varphi(0) = 0$$

there are ξ_0 and ξ_1 such that $0 < \xi_0 < \xi_1 < T$ and $\varphi'(\xi_0)\cdot\varphi'(\xi_1) < 0$. Then for every large enough integer k, $F'((\xi_0 + kT)^{1/\delta})$ has the same sign as $\varphi'(\xi_0)$ and $F'((\xi_1 + kT)^{1/\delta})$ the opposite sign. Moreover, $F'(-\xi) = -F'(\xi)$ and this ensures conditions (iii) and (iv).

Finally, if $0 \leqslant \xi_2 \leqslant T$ and $\eta = \varphi(\xi_2) > 0$, we have, for $t_k = (\xi_2 + kT)^{1/\delta}$,

$$F(t_k) = t_k^q \varphi(\xi_2) = \eta t_k^q$$

and condition (ii′) follows since $|F(\xi)| \leqslant \|\varphi\|_\infty \cdot |\xi|^q$. ∎

In particular, for $n \leqslant 5$ and $\delta = 1$, we get the following.

EXAMPLE 3.10. If φ is a nonconstant $\mathscr{C}^1$ periodic function taking positive values and $2 < q < \frac{n+2}{n-2}$ then the function

$$F\colon \xi \mapsto |\xi|^q\,\varphi(\xi)$$

satisfies the conditions of Corollary 3.7.

If one wants criteria related on the continuous function $f = F'$, one gets the following

EXAMPLE 3.11. Let $0 < \delta \leqslant 1$ and $\varphi \neq 0$ a be a continuous periodic function with null mean on a period. If $1 + \delta < p < \frac{n+2}{n-2}$, the function defined by

$$F(\xi) = \int_0^\xi |t|^p\,\varphi(|t|^\delta)\,dt$$

satisfies the conditions of Corollary 3.7.

Proof. Let T be the period of φ. Since $\varphi \neq 0$ and $\int_0^T \varphi(t)\,dt = 0$, the function φ takes on each period positive and negative values. And this ensures conditions (iii) and (iv). Since φ is bounded, condition (i) is satisfied.

Since $\int_0^T \varphi(t)\,dt = 0$, all primitives of φ are periodic and one of them has a null mean on $[0, T]$. This latter is the derivative of a T-periodic function: hence there is a $\mathscr{C}^2$ periodic function ψ on $\mathbb{R}$ such that $\psi'' = \varphi$. Then, for $\xi > 0$,

$$\begin{aligned}
F(\xi) &= \int_0^\xi t^p \psi''(t^\delta)\,dt = \int_0^\xi \frac{t^{p+1-\delta}}{\delta}\,(\delta t^{\delta-1}\psi''(t^\delta))\,dt \\
&= \left[\frac{t^{p+1-\delta}}{\delta}\,\psi'(t^\delta)\right]_0^\xi - \frac{p+1-\delta}{\delta^2}\int_0^\xi t^{p+1-2\delta}(\delta t^{\delta-1}\psi'(t^\delta))\,dt \\
&= \left[\frac{t^{p+1-\delta}}{\delta}\,\psi'(t^\delta) - \frac{p+1-\delta}{\delta^2}\,t^{p+1-2\delta}\psi(t^\delta)\right]_0^\xi \\
&\quad + \frac{p+1-\delta}{\delta^2}\int_0^\xi \frac{d}{dt}\,(t^{p+1-2\delta})\,\psi(t^\delta)\,dt
\end{aligned}$$

and since

$$\int_0^\xi \left|\frac{d}{dt}\,(t^{p+1-2\delta})\,\psi(t^\delta)\,dt\right| \leqslant \|\psi\|_\infty \int_0^\xi \left|\frac{d}{dt}\,(t^{p+1-2\delta})\right|\,dt = \|\psi\|_\infty\,\xi^{p+1-2\delta}$$

we get

$$F(\xi) = \frac{\xi^{p+1-\delta}}{\delta}\,(\psi'(\xi^\delta) + O(\xi^{-\delta})).$$

Since ψ' is bounded and F is odd this ensures the first part of condition (ii′) with $q = p+1-\delta$. And since ψ' is periodic, not zero, and satisfies $\int_0^T \psi'(t)\,dt = 0$, there is some $t^* \in [0, T]$ such that $\psi'(t^*) > 0$. Then, for $t_k = (t^*, kT)^{1/8}$,

$$\limsup_{\xi\to+\infty}\, \xi^{-q}F(\xi) \geqslant \lim_{k\to\infty}\, t_k^{-p-1+\delta}F(t_k) = \frac{\psi'(t^*)}{\delta} > 0$$

which ensures the last part of condition (ii′). ∎

In particular, for $n \leqslant 5$ and $\delta = 1$, we get the following

Example 3.12. Let φ be a continuous T-periodic function on $\mathbb{R}$ with null mean on $[0, T]$. If $2 < p < \frac{n+2}{n-2}$, the function

$$F(\xi) = \int_0^\xi |t|^p\,\varphi(t)\,dt$$

satisfies the hypotheses of Corollary 3.7.

Notice that condition (iii) of Theorem 3.1 and condition (ii) of Theorem 2.3 are incompatible. Thus it would be an interesting question to know whether there exists functions f satisfying condition (iii) of Theorem 3.1 and for which we have $\lim_{r \to +\infty} Q(r) = +\infty$. But this seems to be quite difficult since the behavior of the function Q at infinity is not easy to derive from that of the function F.

4. GENERICITY OF FUNCTIONS OF TYPE LM_∞

For the oscillating functions we looked at in the previous section, the functional Φ_λ has for each $\lambda > 0$ unboundedly many local maxima. Nevertheless these functions are not necessarily of type LM_∞ since we do not have any control on the function Q. Although it is probably hard to produce simple explicit examples for $n \geqslant 2$, we shall show that in some sense almost all function f is of type LM_∞.

Assume $p < \frac{n+2}{n-2}$ and define the Banach space of continuous functions

$$E_p = \left\{ f \in \mathscr{C}(\mathbb{R}) : \lim_{t \to \infty} \frac{|f(t)|}{1+|t|^p} = 0 \right\}$$

equipped with the norm $|||f||| = \sup_{t \in \mathbb{R}} (|f(t)|/(1+|t|^p))$ which is isomorphic to the space $\mathscr{C}_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ vanishing at infinity.

THEOREM 4.1. *The subset $\mathscr{M}_p$ of E_p consisting of the elements of type LM_∞ is a residual set in E_p.*

Define for $m \in \mathbb{N}$ the subsets U_m and V_m of E_p by

$$U_m = \left\{ f \in E_p : \exists r > m \quad \mu_f(r) < \frac{1}{m} r^2 \right\}$$

$$V_m = \{ f \in E_p : \exists r > m \quad \mu_f(r) > m r^2 \},$$

where $\mu_f(r) = \sup_{\|u\| = r} J_f(u)$ and $J_f(u)$ denotes, as at the beginning of this paper,

$$J_f(u) = \int_\Omega \left(\int_0^{u(x)} f(s)\, ds \right) dx.$$

LEMMA 4.2. *The sets U_m and V_m are open in E_p.*

Proof. The function $f \mapsto J_f(u)$ is clearly linear for each $u \in H_0^1(\Omega)$ from E_p to $\mathbb{R}$, and since $p+1 < \frac{2n}{n-2}$ there is a constant γ such that $\|u\|_{p+1} \leqslant \gamma \|u\|$ for each $u \in H_0^1(\Omega)$. We have by an easy computation:

$$\left|\int_0^\xi f(s)\,ds\right| \leqslant |||f||| \int_0^{|\xi|} (1+|s|^p)\,ds = |||f||| \left(|\xi| + \frac{1}{p+1}|\xi|^{p+1}\right)$$
$$\leqslant |||f|||\,(1+|\xi|^{p+1}).$$

Hence

$$|J_f(u)| \leqslant |||f||| \int_\Omega (1+|u(x)|^{p+1})\,dx \leqslant |||f|||\,(|\Omega| + \gamma^{p+1}\|u\|^{p+1}),$$

from which we deduce:

$$|\mu_f(r) - \mu_g(r)| \leqslant \sup_{\|u\|=r} |J_f(u) - J_g(u)| \leqslant |||f-g|||\,(|\Omega| + \gamma^{p+1} r^{p+1}).$$

Thus for each r the function $f \mapsto \mu_f(r)$ is continuous, and it follows that the sets

$$U_m = \bigcup_{r>m} \left\{ f \in E_p : \mu_f(r) < \frac{1}{m} r^2 \right\}$$

and

$$V_m = \bigcup_{r>m} \{ f \in E_p : \mu_f(r) > mr^2 \}$$

are open in E_p. ∎

Lemma 4.3. *The sets U_m are dense in E_p.*

Proof. Let $f_0 \in E_p$ and $\varepsilon > 0$. Since the set $\mathscr{K}(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with compact support is dense in $\mathscr{C}_0(\mathbb{R})$, there is $g_1 \in \mathscr{K}(\mathbb{R})$ such that

$$\sup_{t \in \mathbb{R}} \left| g_1(t) - \frac{f_0(t)}{1+|t|^p} \right| < \varepsilon.$$

Hence, with $f_1(t) = (1+|t|^p)\,g_1(t)$, $f_1 \in \mathscr{K}(\mathbb{R})$ and $|||f_1 - f_0||| < \varepsilon$. Moreover, for every $\xi \in \mathbb{R}$, we have

$$\left|\int_0^\xi f_1(s)\,ds\right| \leqslant \|f_1\|_1 = \int_{-\infty}^{+\infty} |f_1(s)|\,ds < +\infty.$$

It follows that for each $u \in H_0^1(\Omega)$:

$$J_f(u) = \int_\Omega \left(\int_0^{u(x)} f_1(s)\, ds \right) dx \leqslant \int_\Omega \|f_1\|_1\, dx = \|f_1\|_1\, |\Omega|.$$

Hence

$$\mu_f(r) \leqslant \|f_1\|_1\, |\Omega| - \lambda r^2.$$

Thus $f_1 \in U_m$ for every m, and $U_m \cap B(f_0, \varepsilon) \neq \varnothing$ for each $m \in \mathbb{N}$ and each $f_0 \in E_p$. ▮

LEMMA 4.4. *The sets V_m are dense in E_p.*

Proof. Let $f_0 \in E_p$ and $\varepsilon > 0$. As above, there exists a $f_1 \in \mathscr{K}(\mathbb{R})$ such that $|||f_0 - f_1||| < \frac{\varepsilon}{2}$. Let q be such that $1 < q < p$. Then the function f^* defined by

$$f^*(t) = t \cdot |t|^{q-1}$$

belongs to E_p and we have $|||f^*||| = \sup_{t>0} \frac{t^q}{1+t^p}$.

For $t \leqslant 1$, we have $t^q \leqslant 1 \leqslant 1 + t^p$; hence $\frac{t^q}{1+t^p} \leqslant 1$. And for $t \geqslant 1$, we have $t^q \leqslant t^p \leqslant 1 + t^p$; hence $\frac{t^q}{1+t^p} \leqslant 1$. We conclude that $|||f^*||| \leqslant 1$. So $f_2 = f_1 + \frac{\varepsilon}{2} f^* \in B(f_0, \varepsilon)$. As in the lemma above, we have:

$$\left| \int_0^\xi f_1(t)\, dt \right| \leqslant \int_{-\infty}^{+\infty} |f_1(t)|\, dt = \|f_1\|_1.$$

Thus

$$\left| \int_0^\xi f_2(t)\, dt \right| \geqslant \frac{\varepsilon}{2} \int_0^\xi f^*(t)\, dt - \left| \int_0^\xi f_1(t)\, dt \right| \geqslant \frac{\varepsilon}{2(q+1)} |\xi|^{q+1} - \|f_1\|_1,$$

and it follows that

$$J_{f_2}(u) = \int_\Omega \left(\int_0^{u(x)} f_2(t)\, dt \right) dx \geqslant \frac{\varepsilon}{2(q+1)} \int_\Omega |u(x)|^{q+1}\, dx - \|f_1\|_1\, |\Omega|$$

$$\geqslant \frac{\varepsilon}{2(q+1)} \|u\|_{q+1}^{q+1} - \|f_1\|_1\, |\Omega|.$$

If we choose some u_0 in $H_0^1(\Omega)$ such that $\|u_0\| = 1$, we get

$$\mu_{f_2}(r) - mr^2 \geqslant J_{f_2}(r \cdot u_0) - mr^2 \geqslant \frac{\varepsilon r^{q+1}}{2(q+1)} \|u_0\|_{q+1}^{q+1} - \|f_1\|_1\, |\Omega| - mr^2$$

and since $q+1>2$ we can easily find a $r>m$ such that $\|f_1\|_1\,|\Omega|+mr^2<((\varepsilon r^{q+1})/(2(q+1)))\,\|u_0\|_{q+1}^{q+1}$. This shows that $f_2\in V_m$ and thus that $V_m\cap B(f_0,\varepsilon)\neq\varnothing$. And the proof of the lemma is complete. ∎

Proof of Theorem 4.1. The set

$$G=\bigcap_m U_m\cap\bigcap_m V_m$$

is the intersection of a countable family of dense open sets; thus it is comeager in the Banach space E_p and hence dense. We shall prove that $G\subset\mathscr{M}_p$.

Let $f\in G$. We can choose an increasing sequence $(r_m)_{m\geqslant 1}$ such that $r_m\geqslant m$ and

$$\mu_f(r_{2m-1})<\frac{1}{2m-1}\,r_{2m-1}^2$$

$$\mu_f(r_{2m})>2m\cdot r_{2m}^2.$$

Then

$$\begin{aligned}Q(r_{2m+1})&=\inf_{s<r_{2m+1}}\frac{\mu(r_{2m+1})-\mu(s)}{r_{2m+1}^2-s^2}\leqslant\frac{\mu(r_{2m+1})-\mu(r_{2m})}{r_{2m+1}^2-r_{2m}^2}\\&\leqslant\frac{1}{r_{2m+1}^2-r_{2m}^2}\left(\frac{r_{2m+1}^2}{2m+1}-2m\cdot r_{2m}^2\right)\\&\leqslant\frac{1}{r_{2m+1}^2-r_{2m}^2}\left(\frac{r_{2m+1}^2}{2m+1}-\frac{r_{2m}^2}{2m+1}\right)\\&\leqslant\frac{1}{2m+1}.\end{aligned}$$

And this shows that $\liminf_{r\to\infty}Q(r)\leqslant 0$. Moreover Φ_{2m-1} attains its maximum on $B(r_{2m+1})$ at a point u_m, and we have

$$\begin{aligned}\Phi_{2m-1}(u_m)&=\sup_{u\in B(r_{2m+1})}\Phi_{2m-1}(u)\geqslant\sup_{\|u\|=r_{2m}}\Phi_{2m-1}(u)\\&=\mu_f(r_{2m})-(2m-1)\,r_{2m}^2\geqslant r_{2m}^2>0\end{aligned}$$

while for each $u\in S(r_{2m+1})$ we have

$$\Phi_{2m-1}(u)\leqslant\mu_f(r_{2m+1})-(2m-1)\,r_{2m+1}^2<\left(\frac{1}{2m+1}-2m\right)r_{2m}^2<0.$$

Thus u_m lies in the interior of the ball $B(r_{2m+1})$ and is a critical point of Φ_λ. Moreover, since $S(r_{2m-1})$ is weakly dense in $B(r_{2m-1})$ we have

$$\sup_{\|u\| \leqslant r_{2m-1}} J_f(u) = \sup_{\|u\| = r_{2m-1}} J_f(u) = \mu_f(r_{2m-1}) < \frac{1}{2m-1} r_{2m-1}^2 < \frac{1}{2m-1} r_{2m}^2.$$

Hence, for $\|u\| \leqslant r_{2m-1}$

$$\Phi_{2m-1}(u) \leqslant J_f(u) < \frac{1}{2m-1} r_{2m}^2 \leqslant r_{2m}^2 \leqslant \Phi_{2m-1}(u_m)$$

and this implies that $\|u_m\| > r_{2m-1} \geqslant 2m-1$. Finally, since Φ_{2m-1} attains its maximum over $B(\|u_m\|)$ on $S(\|u_m\|)$, we have $Q(\|u_m\|) \geqslant 2m-1$ by Lemma 2.1.

So $\limsup_{r\to\infty} Q(r) = +\infty$, and we are done. ∎

Remark 4.5. We could also replace the Banach space E_p by

$$\tilde{E}_p := \left\{ f \in \mathscr{C}(\mathbb{R}) := \sup_{t\in\mathbb{R}} \frac{|f(t)|}{1+|t|^p} < +\infty \right\}$$

equipped with the same norm $|||f||| := \sup_{t\in\mathbb{R}}(|f(t)|/(1+|t|^p))$. Then $\tilde{E}_p$ is a Banach space isomorphic to the space $\mathscr{C}_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$. But in this case the previous result is not longer true: the set $\tilde{\mathscr{M}}_p$ of functions of type LM_∞ is not dense in $\tilde{E}_p$.

Indeed, if f^* denotes the function: $t \mapsto t\cdot|t|^{p-1}$ which belongs to $\tilde{E}_p$, and $\rho < |||f^*||| = 1$, for every function f of $\tilde{E}_p$ satisfying $|||f - f^*||| < \rho$ we will have for every $t \in \mathbb{R}$:

$$t\cdot f(t) \geqslant (1-\rho)\,|t|^{p+1} - \rho|t|$$
$$|f(t)| \leqslant (1+\rho)\,|t|^p + \rho.$$

Then it follows from Theorem 2.3 that

$$\liminf_{r\to\infty} Q_f(r) = +\infty$$

and hence that f is not of type LM_∞. So $\tilde{\mathscr{M}}_p \cap B(f^*, \rho) = \varnothing$; that is f^* does not belong to the closure in $\tilde{E}_p$ of the set $\tilde{\mathscr{M}}_p$.

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